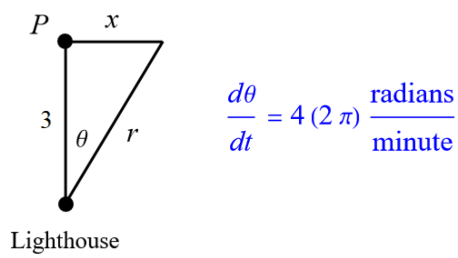
Exercise 44

A lighthouse is located on a small island 3 km away from the nearest point P on a straight shoreline and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from P?

Solution

Draw a schematic of the lighthouse's beam at a certain time.



The aim is to find dx/dt when $x = 1$. Use a trigonometric function to relate the angle θ with convenient sides of the triangle.

$$\tan\theta = \frac{x}{3}$$

Solve for x.

$$x = 3\tan\theta$$

Take the derivative of both sides with respect to time by using the chain rule.

$$\frac{d}{dt}(x) = \frac{d}{dt}(3\tan\theta)$$

$$\begin{aligned}\frac{dx}{dt} &= (3\sec^2\theta)\cdot\frac{d\theta}{dt}\\ &= 3\left(\frac{r}{3}\right)^2\cdot 4(2\pi)\\ &= 3\left(\frac{r^2}{9}\right)\cdot 8\pi\\ &= 3\left(\frac{x^2+9}{9}\right)\cdot 8\pi\\ &= \frac{x^2+9}{3}(8\pi)\end{aligned}$$

Therefore, when the beam of light is 1 km from P, the beam is moving at

$$\left.\frac{dx}{dt}\right|_{x=1} = \frac{(1)^2+9}{3}(8\pi) = \frac{80\pi}{3}\ \frac{\text{km}}{\text{min}} \approx 83.7758\ \frac{\text{km}}{\text{min}}.$$